J. Fluid Mech. (1968), *vol.* 31, *part* 4, *pp.* 819–825
Printed in Great Britain

The transient for Stokes's oscillating plate: a solution in terms of tabulated functions

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(Received 4 August 1967)

The motion of a semi-infinite incompressible fluid caused by the sinusoidal oscillation of a plane flat plate is termed Stokes's problem. When the plate starts from rest in a still fluid a transient solution must be added to Stokes's well-known steady-state result. This paper presents a closed-form expression for the transient solution. Previous answers have contained a non-standard integral which could not be evaluated. The answer presented herein contains exponentials and error functions of a complex argument. These functions are readily available in newer mathematical tables. Graphs of the transient solution are presented for both $\sin(T)$ and $-\cos(T)$ boundary conditions. Velocity distributions in the fluid are also plotted and it is found that the transient period is essentially complete in one-half cycle for the cosine oscillation and in a full cycle for the sine wave case.

1. Introduction

Stokes's result for an oscillating plate, called Stokes's second problem by Schlichting, is a well-known fundamental solution in viscous fluid mechanics. The physical problem consists of a semi-infinite incompressible fluid bounded by a flat plate. The plate oscillates in its own plane with a velocity $u(0, t) = u_0 \sin(\omega t)$. Stokes's result is the steady-state solution which applies after the effect of any initial velocity profile has died out. The topic of this paper is the 'transient solution' where the fluid is assumed initially at rest. This solution must be added to Stokes's solution to obtain the complete answer, including the initial condition.

Stokes's solution is not only of fundamental theoretical interest but it also occurs in many contexts in applied problems. It arises in acoustic streaming around an oscillating body. Another example is an established boundary layer with an imposed fluctuation in the free-stream velocity. In this problem Stokes's result appears as a perturbation in the high frequency limit. Since the flow is incompressible it is immaterial whether the plate oscillates in a stagnant fluid or the plate is fixed and the fluid oscillates. The transient solution is also important in certain unsteady boundary layers; those starting from rest or those with an external velocity having an arbitrary time dependence (the region of validity is restricted to be away from the influence of a leading edge). Because the problem is linear, it is useful to replace the arbitrary velocity history by a Fourier series. Each term in the series then yields a Stokes's problem where both the transient and steady-state components are significant.

There is an analogy between viscous diffusion problems and unsteady heat conduction problems. Hence problems in one field may be directly taken over into the other. The solution here for Stokes's transient can be reinterpreted as applying to the sinusoidal heating of a semi-infinite wall which is initially at a uniform temperature. Again the more general problem where the wall is heated in an arbitrary manner can be decomposed into a sequence of Stokes's problems using Fourier series.

To the writer's knowledge, a closed form solution to the transient problem has not been given before. Carslaw & Jaeger (1959, p. 64), the standard source book for heat conduction, does not discuss this problem. Schlichting (1960, p. 76) gives the following answer in the form of an integral which he attributes to Müller,

$$\frac{u^t}{u_0} = \frac{2\nu\omega}{\pi}\int_0^\infty \frac{\zeta\cos(\zeta y)}{\omega^2+\nu^2\zeta^4}\exp(-\nu\zeta^2 t)\,d\zeta. \tag{1}$$

In this equation u^t is the transient velocity component, t time, y the distance perpendicular to the plate, ν the kinematic viscosity; u_0 and ω are the amplitude and frequency of the oscillation respectively. Arpaci (1966, pp. 423 and 280) recently attempted the solution using Fourier transforms; however, he could not perform the inversion and also left the answer as an unevaluated integral. The purpose of this paper is to present a solution in terms of standard mathematical functions.

2. Mathematical formulation and solution

The fluid is taken to occupy the upper half-plane with the plate on the x-axis. The velocity satisfies the diffusion equation and boundary conditions:

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial y^2} = 0 \quad (y,t>0),$$

$$u(y,0) = 0,$$

$$u(\infty,t) < \infty,$$

$$u(0,t) = u_0\sin(\omega t).$$

Capital letters will denote nondimensional variables which are defined by

$$U = u/u_0, \quad T = \omega t, \quad Y = y(\omega/\nu)^{\frac{1}{2}}.$$

The problem now reads

$$\frac{\partial U}{\partial T} - \frac{\partial^2 U}{\partial Y^2} = 0 \quad (Y,T>0), \tag{2}$$

$$U(Y,0) = 0, \tag{3}$$

$$U(\infty,T) < \infty, \tag{4}$$

$$U(0,T) = \sin(T). \tag{5}$$

The velocity may be decomposed into a steady state and a transient component

$$U = U^s + U^t. \tag{6}$$

Both components satisfy the same form of differential equation, (2). The steady-state component is given in any of the references cited above.

$$U^s = \exp(-Y/\surd 2)\sin(T - Y/\surd 2). \tag{7}$$

This solution satisfies boundary conditions (3) and (4) but not the initial condition (2). If the transient solution satisfies the boundary conditions

$$U^t(Y,0) = -\exp(-Y/\surd 2)\sin(-Y/\surd 2), \tag{8}$$

$$U^t(\infty, T) < \infty, \tag{9}$$

$$U^t(0,T) = 0, \tag{10}$$

then the composition of the transient and steady-state solutions will completely satisfy equations (2) to (5).

The general transient problem can be solved by integrating the following expression, given by Carslaw & Jaeger,

$$U^t = \frac{1}{2(\pi T)^{\frac{1}{2}}}\int_0^\infty f(\zeta)\{\exp[-(Y-\zeta)^2/4T] - \exp[-(Y+\zeta)^2/4T]\}\,d\zeta. \tag{11}$$

The function $f(\zeta)$ is the initial profile, in our case equation (8). Substitution of (8) into (11) does not lend to a tractable integral, as the experience of other authors points out. Instead of (8), an equivalent expression using the notation of complex variables will be used,

$$U^t(Y,0) = \operatorname{Im}\exp[-(1-i)\,Y/\surd 2]. \tag{12}$$

Substitution of (12) into (11) gives integrals which can be evaluated by standard techniques. The result for the transient velocity component is

$$\begin{aligned} U^t(Y,T) = \operatorname{Im}\{&-\tfrac{1}{2}\exp(CY/\surd 2 - iT)\operatorname{erfc}[(\tfrac{1}{2}T)^{\frac{1}{2}}(C + Y/(T\surd 2))] \\ &+ \tfrac{1}{2}\exp(-CY/\surd 2 - iT)\operatorname{erfc}[(\tfrac{1}{2}T)^{\frac{1}{2}}(C - Y/(T\surd 2))]\}. \end{aligned} \tag{13}$$

In this equation, C is the complex constant $C = 1-i$. It is noted that each term in (13) satisfies the differential equation separately.

We can now understand the difficulties associated with previous attempts to integrate equation (1). To obtain the answer in real variables, equation (13) should be separated into its real and imaginary parts. This could be done if there were an expression for the error function,

$$\operatorname{erf}(x+iy) = F(x,y) + iG(x,y), \tag{14}$$

where F and G contain elementary functions. Salzer (1951) (see also Abramowitz & Stegun 1964, p. 297) gives an infinite series of the form (14) which *approximates* the error function but it appears that no exact relation exists.

Equation (13) also contains the answer to a related problem. It can be verified that the real part of (13) is the answer to the transient problem with the plate oscillating as $-\cos T$ instead of $\sin T$ in (5).

As it stands now the answer (13) contains the error function of a complex argument. The tables given in Abramowitz & Stegun (1964) do not give $\operatorname{erf}(z)$

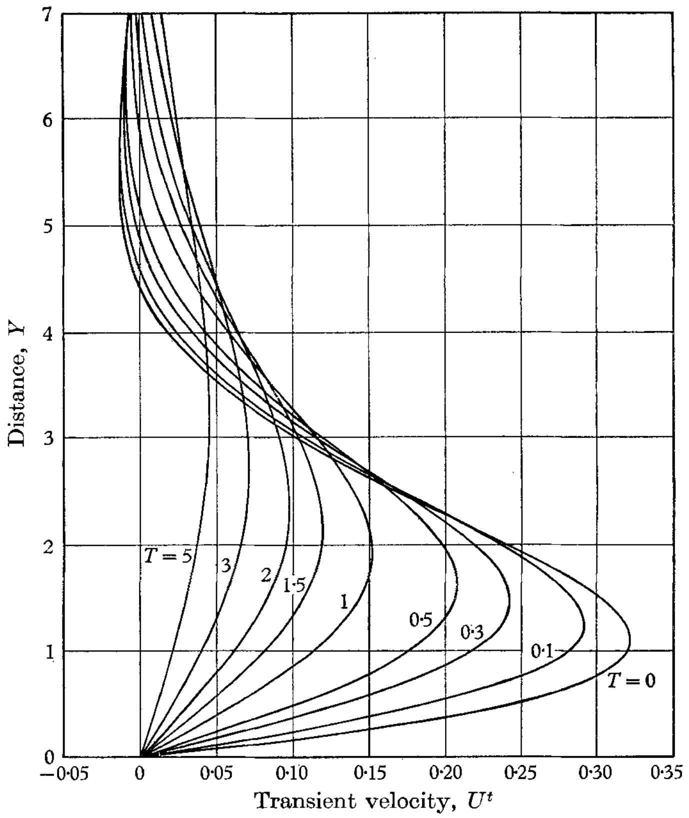

FIGURE 1. Transient velocity distribution, plate velocity $\sin(T)$.

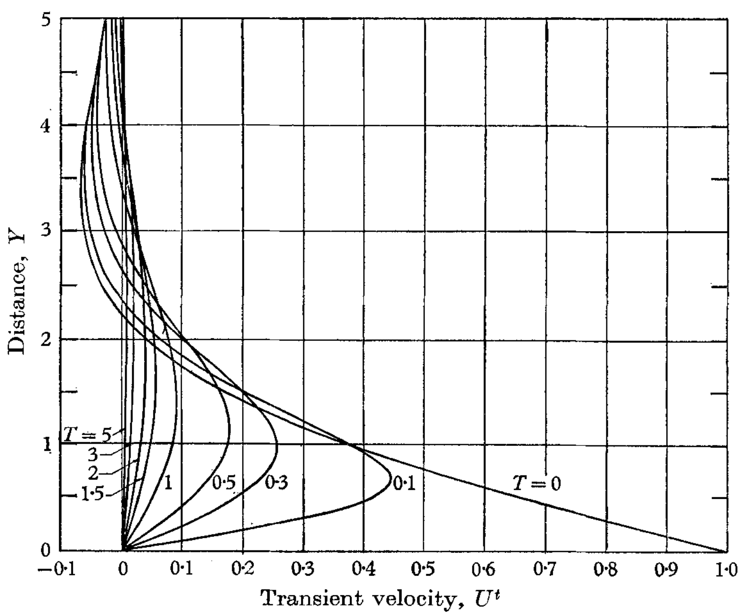

FIGURE 2. Transient velocity distribution, plate velocity $-\cos(T)$.

directly but an auxiliary function. The auxiliary function is used to give a smoother function which allows accurate interpolation. The function is defined as

$$w(z) = \exp(-z^2)\,\mathrm{erfc}(-iz). \tag{15}$$

In terms of (15) the answer has a compact form and may be found directly in the tables

$$U^t = \tfrac{1}{2}\exp[-Y^2/(4T)]\,.\,[w(z_1) - w(z_2)], \tag{16}$$

where

$$z_1 = (\tfrac{1}{2}T)^{\frac{1}{2}} + i[(\tfrac{1}{2}T)^{\frac{1}{2}} - Y/(2\sqrt{T})],$$

$$z_2 = (\tfrac{1}{2}T)^{\frac{1}{2}} + i[(\tfrac{1}{2}T)^{\frac{1}{2}} + Y/(2\sqrt{T})].$$

The imaginary part of (16) applies when the plate motion is $\sin(T)$ and the real part for $-\cos(T)$. Graphs of equation (16) are given in figures 1 and 2.

3. Characteristics of the solution

It is easy to see that the transient dies out rapidly because of the exponential in equation (1). By plotting the transient solution we can get an idea of just how rapidly this happens. The transient component of the velocity for the case when the wall velocity varies as $\sin(T)$ is given in figure 1. The maximum velocity on figure 1 is slightly greater than 0·3 and occurs near $Y = 1$. The curves decay rapidly and the maximum point moves further into the fluid. At a time $T = 5$, slightly less than a full cycle, the maximum velocity has been reduced to less than 0·05.

Figure 2 displays the transient velocity for the second case; that is, when the plate oscillates as $-\cos(T)$. This problem has a discontinuity in the boundary conditions at the plate. Initially the plate velocity is one, but jumps to zero for $T > 0$. The scales on figures 1 and 2 are different, since the maximum velocity is now one. Again the position of the maximum moves into the fluid as time progresses. Although the initial velocity is much higher in this case the decay occurs more rapidly; all velocities are less than 0·05 at a time $T = 2$. The time for this to occur in the first case was $T = 5$.

The complete starting problem velocity profiles are shown in the next two figures. Velocity profiles are given in figure 3 for the case where the fluid is initially still and the plate is moved so that the velocity varies as $\sin(T)$. For comparison the steady-state solutions are plotted as dashed lines. The first curve at $T = 0{\cdot}3$ shows the viscous 'wave' has penetrated only slightly into the fluid. The corresponding steady-state velocity profile shows that most of the fluid has a negative velocity, the result of the steady-state plate velocity just completing its negative half cycle. The next curve at $T = 1{\cdot}5$ shows a deeper penetration and a decay in the difference between the starting and the steady-state solutions. This difference, which is the transient solution, is seen from figure 1 to be a maximum of 0·12 at $Y = 2$. The plate velocity reverses and becomes negative for $T = 3{\cdot}5$. A maximum now occurs in the interior of the fluid and propagates inward. Near the bottom of the oscillation, $T = 5$, the transient and steady-state curves are almost the same; as noted previously, the difference has decayed to less than 0·05.

The starting phase for a fluid with a plate velocity $-\cos(T)$ is plotted on figure 4. Again the dashed lines give the corresponding steady-state distribution. The plate velocity which is initially zero jumps to -1 and the fluid begins to respond. The plate velocity begins to rise and the difference between the starting and steady state rapidly decreases until they are essentially the same at $T = 2{\cdot}5$.

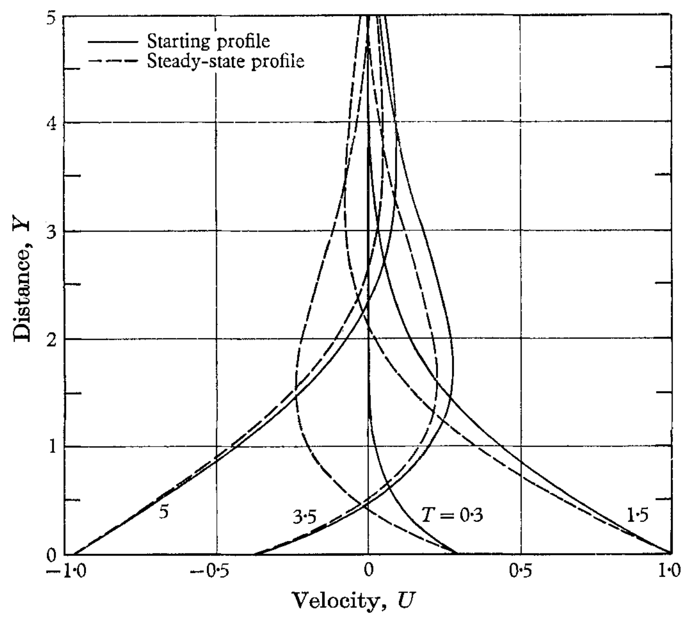

FIGURE 3. Starting phase velocity profiles, plate velocity $\sin(T)$.

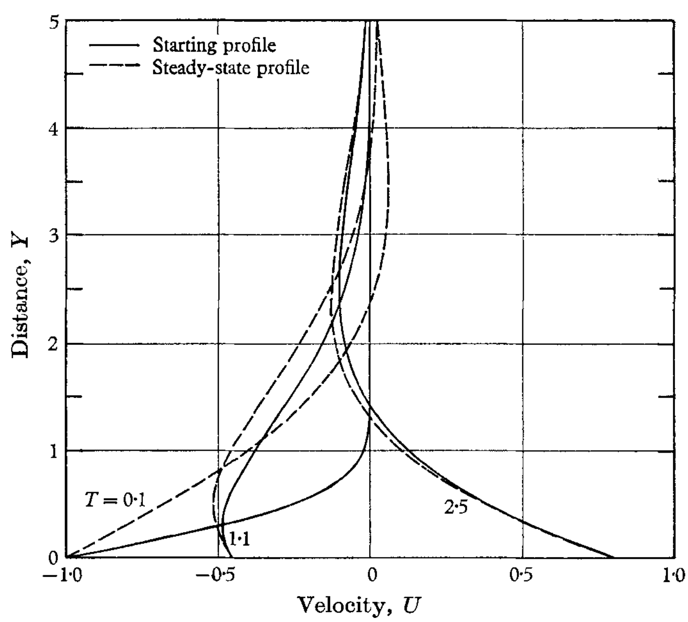

FIGURE 4. Starting phase velocity profiles, plate velocity $-\cos(T)$.

In general, the transient effect will be negligible in less than one-half cycle for the cos (T) surface temperature. The sin (T) temperature oscillation requires a little longer for the transient to decay but it still becomes negligible in less than a full cycle.

4. Summary

A closed form solution to the transient component of Stokes's problem was presented. The real part of the complex result corresponds to a plate oscillation of $-\cos(T)$ while the imaginary part is associated with sin (T). The error function of a complex variable appears in the answer and does not allow the solution to be split into real and imaginary parts consisting of elementary functions. Nevertheless, numerical values can be easily calculated, since the complex error function, or its equivalent, is given in newer mathematical tables.

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